Journal of Engineering Mathematics **39:** 79–86, 2001.

Fast computation of limit cycles in an industrial application

SHAY GUERON and NADAV LIRON[1]
Department of Mathematics, University of Haifa, Haifa, 31905, Israel
[1]*Department of Mathematics, Technion – I.I.T, Haifa, 32000, Israel*

Received 6 December 1999; accepted in revised form 4 July 2000

Abstract. A model problem is described that requires the study of a system of the form $\dot{\mathbf{v}}(t) = \varepsilon \mathbf{F_P}(\mathbf{v(t)}, \mathbf{t})$ which depends on a set of parameters **P**, and where $\varepsilon \ll 1$. The problem comes from an industrial application where it is a kernel of an optimization procedure. The optimization depends on computing the limit cycle, and the problem needs to be solved repeatedly. Short computation time is therefore essential. The naive approach is to integrate the equation forward in time, starting from an arbitrary initial condition, until the transients disappear and the limit cycle is approximated within a given tolerance. This approach is too slow and thus impractical in the context of the optimization procedure. The problem involves two types of asymptotic considerations: long-time asymptotics and small-parameter asymptotics. Here a simple approach is demonstrated, based on implementing the averaging method. This reduces the solution time to the point that the optimization procedure becomes feasible.

Key words: limit cycle, dynamical systems, Poincaré-Bendixson

1. Motivation

Our study is motivated by an industrial engineering problem of designing a piezoelectric motor. The design stage of a certain component of the product required the solution of its model which is represented by a system of the form $\dot{\mathbf{v}}(t) = \varepsilon \mathbf{F_P}(\mathbf{v(t)}, \mathbf{t})$. This was the kernel of an optimization procedure. Here, **v** is an unknown m-dimensional vector modeling velocity, **F** is the given m-dimensional right-hand side (RHS) of the system, modeling acceleration. The notation $\mathbf{F_P}$ indicates that the problem depends on a set of parameters denoted by **P**. The RHS was continuous and piecewise smooth, typically bounded between two linear functions, and also periodic in t. The periodicity accounts for a periodic forcing component in the system, and the nonsmoothness accounts for the changes in the direction of this force with respect to other forces such as friction. The small parameter ε appears, due to the fast oscillations in the system, by rescaling the time ($\omega\tau \to t$).

Based on practical arguments, the physical system as well as the solution of its model were assumed to approach some periodic behavior for large enough t. In mathematical terms, this implies that there exists a stable (attracting) limit cycle. The design stages required the approximation of the limit cycle and some of its functionals such as for example, the average over one period. An analytic solution was not always available and a numerical solution was attempted. The approach adopted by the R&D team in that industry was to choose zero initial conditions for the velocity (as in a real experimental situation), to solve the system numerically until all transients have disappeared, to obtain the limit cycle numerically, and to compute the required functionals. The problem with this approach was (a) the large number of parameters (over ten in this case) on which the system depended required a great deal of runs, (b) the long time before the transients disappeared for each run slowed down the study of the model so

much so as to hinder the time-table set by the industry for the development of the project. The speed of approach to the limit cycle depends crucially on the initial guess.

The approach we describe here takes advantage of asymptotics in two ways.

(i) We use 'small ε' perturbation to obtain an initial guess for the entire computation

(ii) We consider the solution at time intervals of the period of the limit cycle as values in a fixed-point iteration process and use acceleration techniques to rapidly approach a point on the limit cycle.

The combination of these two techniques indeed solved the problem practically.

The method we describe was successfully applied to systems with dimensionality $m > 10$. In this note we demonstrate the study of a one dimensional ($m = 1$) system, where we can easily prove that the solution converges to a limit cycle, and the study of a two dimensional ($m = 2$) system.

2. A case study

Consider the one dimensional ordinary differential equation (ODE)

$$\dot{v}(t) = \varepsilon F_{\mathbf{P}}(v(t), t), \tag{1}$$

where $(\dot{})$ denotes differentiation with respect to time t and ε is a small parameter. The notation $F_{\mathbf{P}}$ indicates that the right-hand side of Equation (1) depends on a set of parameters, namely $\mathbf{P}$. For simplicity, we omit the explicit notation of this parameter dependence, and hereafter write $F(v, t) = F_{\mathbf{P}}$. The function $F(v, t)$ is continuous and piecewise smooth. Suppose also that $F(v, t)$ is periodic in the variable t, and denote its period by T.

Equations and systems of the type (1) with smooth and nonsmooth right-hand side are an important class of dynamical systems that have various applications in different fields. In modeling and design applications Equation (1) is studied with different parameter values, and must therefore be repeatedly solved many times (see, for example, [1], [2]).

We assume here that, if the initial condition is chosen within some known interval, the resulting solution of (1) is bounded in $t \in [0, \infty)$. This is a natural assumption in the context of the real applications, where $v(t)$ is interpreted as the velocity of a modeled object, and (1) describes its acceleration. As we explain below, this guarantees the existence of an attracting limit cycle, that is, that $v(t)$ approaches a certain periodic solution (with period T) as $t \to \infty$. We are interested in the following computational problems:

PROBLEM 1. *Compute the limit cycle to which the solution $v(t)$ approaches as $t \to \infty$.*

PROBLEM 2. *Compute the average of the limit cycle solution $v(t)$ over one period,*

$$\frac{1}{T}\int_t^{t+T} v(s)\mathrm{d}s, \qquad \text{for } t \to \infty. \tag{2}$$

An analytic solution to (1) is often not available, or cumbersome to write down, manipulate and get insight from, especially for a function of many parameters. Thus, a numerical approach is required. When these computations need to be repeated, reducing the computational effort of this numerical task is imperative. To that end, we try to minimize the required number of evaluations of $F(v, t)$. This is equivalent to attempting to decrease the number of times that we integrate (1) numerically, over a time interval of length T.

2.1. THE STRAIGHTFORWARD NUMERICAL APPROACH

The straightforward approach for solving problems 1 and 2 is to solve (1) numerically. To that end, we start with some initial condition $v(0) = v_0$ and propagate the numerical trajectory $(t, v(t))$ in time, by applying some numerical ODE integrator (for example a Runge-Kutta scheme). We integrate the ODE for a long enough time interval, until the transients disappear, obtaining a discrete set of points (t_i, v_i), $1 \leq i \leq L$, (L is the number of discretization points) covering one cycle.

A practical way to check that transients have indeed disappeared is to compare the numerical trajectory of the n-th cycle to the $n+1$st cycle, and to stop the procedure when the differences are sufficiently small in some norm, for example, when the RMS $= \sqrt{\frac{1}{L}\sum_1^L (v_j^{n+1} - v_j^n)^2}$ falls below a certain prefixed value. This requires the storing of an appropriate number $(2L)$ of trajectory points. For Problem 2 we need to store only the estimated average over the cycles.

We measure the computational cost of this procedure in terms of the number of times that (1) is integrated over a time interval of length T. Obviously, this cost can be reduced by a good guess for v_0.

2.2. THE FIXED-POINT APPROACH

We now consider our computational problem from a different point of view. Suppose that at some time t_1 we have computed the numerical trajectory point (t_1, v_1), which is a point on the limit cycle. Since $F(v, t)$ is periodic in t with period T, this implies that a) The period of the limit cycle is also T; b) If we choose the initial condition (t_1, v_1), and propagate (1) in time up to $t_2 = t_1 + T$, the computed trajectory point (t_2, v_2) must be equal to the point (t_1, v_1). This motivates the following stability assumption: if point (t_1, v_1), is close to a point on the limit cycle, and we propagate (1) to $t_2 = t_1 + T$, using initial condition (t_1, v_1), then the computed trajectory point (t_2, v_2) is close to the point (t_1, v_1).

Formally, we now define the following map $M : R \to R$; $M = M_t(u^*)$ is the value of the solution of (1) at $t + T$, *i.e.*, $v(t + T)$, given initial condition $v(t) = u^*$.

In these terms, our problem is restructured to the problem of computing the fixed point of the map M. Indeed, once the fixed point is computed (estimated), we can solve problems 1 and 2 at the cost of one additional integration of (1) over one cycle of length T.

Clearly, any numerical ODE integration can be interpreted as an iterative method for computing the fixed point of M, namely:

$$v_{n+1} = M(v_n), \qquad n = 0, 1, \ldots. \tag{3}$$

Since t is arbitrary, we can take any large t and shift it to, say $t = 0$. Thus, the explicit dependence of Equation (3) on the chosen time t is omitted.

We now make the following observations that lead to a practical solution of the problem:

1. The computational cost of one iteration of (3) is the cost of integrating (1) along a time interval of length T. Thus, we can solve the long-time asymptotic problem by using (3) with some ODE integration method. To that end, we need only to store a few values of $M(v_n)$.
2. For any choice of an ODE integrator to compute M, the iterative scheme (3) can be replaced by any other and possibly faster iterative scheme that does not require the use of the derivative of M, for example the Secant Method. Note that we do not use the derivative

of M because this would require the differentiation of an ODE with respect to its initial condition, which we cannot perform here.

3. For any choice of an ODE integrator to compute M in (3), we can accelerate the convergence by applying some acceleration technique to the iterations. One possible example is applying Aitken's acceleration method,

$$\hat{M}_n = M_n - \frac{(M_{n+1} - M_n)^2}{M_{n+2} - 2M_{n+1} + M_n}.$$

2.3. INITIAL CHOICE FOR $0 < \varepsilon \ll 1$

A crucial ingredient in speeding up the convergence of the iterations, is a good choice of the initial guess v_0. Since $\varepsilon \ll 1$, we can use small-parameter asymptotics and argue that the oscillations of the limit cycle have a small magnitude, and apply some asymptotic technique in order to obtain a good estimate for v_0. The simplest method to obtain a good estimate for v_0 or to obtain a good estimate for the average of v over a cycle is to write $v(t) = \bar{v} + \varepsilon\nu(t) + \ldots$ for some unknown parameter $\bar{v}$ and unknown function $\nu(t)$. Since the limit cycle is periodic with period T, and satisfies (1), the equation for the zero order approximated equation for unknown $\bar{v}$ is

$$\int_0^T F(t, \bar{v})\mathrm{d}t = 0. \tag{4}$$

Equation (4) can be solved numerically.

In the next sections we demonstrate the application of the above points.

3. Proving the existence of a limit cycle

A comprehensive study on the existence and stability of limit cycles for problems of our type can be found for example in [3, Chapters 5 and 7]. However, for dimensionality $m = 1$, and with the specific form of the RHS in mind, we can carry out a simple analysis. If the solution of (1) is uniformly bounded for any initial condition, then the Poincaré–Bendixson theorem guarantees that this solution converges either to a unique limit cycle or to a singular point. Since the RHS of (1) is periodic, there are no singular points. Therefore, the iterations (3) converge to a fixed point of M. We now apply this to our case and state the following:

Suppose that $F(v, t)$ is periodic in the variable t with period T, and that $F(v, t)$ is bounded by two linear functions of v (*e.g.*, as in [2]), that is,

$$-a_1 v + b_1 \le F(t, v) \le -a_2 v + b_2 \tag{5}$$

for some positive constants a_1, a_2, b_1, b_2. Then, there exists a value v_0 for which the solution $v(t)$ of (1) with the initial condition $v(0) = v_0$ satisfies $v(T) = v_0$. Since $F(v, t)$ is periodic with period T, this implies that the solution $v(t)$ is the periodic solution of (1).

Proof:

Denote the solution of (1) with $v(0) = v_0$ by $v(t)$. For any t we have

$$\frac{b_1}{a_1} + \left(\frac{b_1}{a_1} + v_0\right)\mathrm{e}^{-a_1 t} \le v(t) \le \frac{b_2}{a_2} + \left(\frac{b_2}{a_2} + v_0\right)\mathrm{e}^{-a_2 t} \tag{6}$$

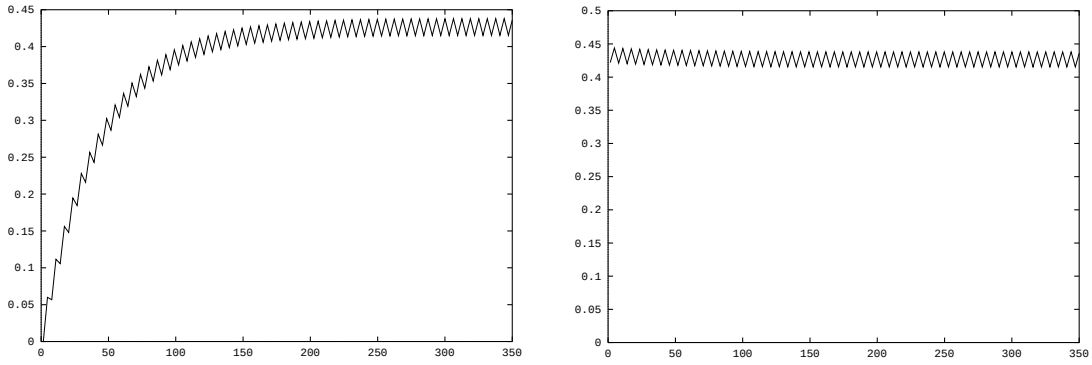

Figure 1. The numerical trajectory for Equation (1). Application of the Runge–Kutta method, $dt = 0{\cdot}01256$, (a) initial condition $v_0 = 0$; (b) initial condition $v_0 = \bar{v} = 0{\cdot}43333$.

Therefore, $e^{-a_1 T} < \lim_{v_0 \to \infty} \dfrac{v(T)}{v_0} \leq e^{-a_2 T} < 1$.

This implies that there exists a sufficiently large value $A_0 > 0$ such that the map M maps the interval $[-A_0, A_0]$ *into itself*. It then follows that M has a fixed point, which is exactly the required value of v_0.

4. Examples

We first illustrate the method with a simple test case

$$\dot{v}(t) = \varepsilon(\text{sign}\,(v(t) - \delta\cos(\omega t)) - \alpha v(t) + b); \tag{7}$$

We choose the parameter values $\varepsilon = 0{\cdot}01$, $\delta = 1$, $\alpha = 3$, $b = 1$, and $\omega = 1$ (*i.e.*, $T = 2\pi$).

Since the solution v is interpreted as the velocity of an object that starts at rest, the natural choice for initial condition is $v_0 = 0$. We used a Runge-Kutta method of order 4, with 30 points along one interval (*i.e.*, $dt \approx 0{\cdot}01256$) and obtained the numerical trajectory displayed in Figure 1, Panel a.

Clearly, integration over many *cycles* (more than 50) is required before the transients disappear and the trajectory settles down at the limit cycle. Note the small oscillations of the limit cycle. Computations using the numerical trajectory show that the average of $v(t)$ over one cycle is $\bar{v} \approx 0{\cdot}42664$.

We show that it is advantageous to first solve (4) numerically. Using the trapezoidal rule with 50 nodes on $[0, T]$, we obtained the zero order approximation $\bar{v} \approx 0{\cdot}43333$, which is a good approximation. If we are only interested in this average, we are already done. To compute the limit cycle itself we start the same iterations with the initial condition $v_0 = \bar{v} \approx 0{\cdot}43333$. The convergence is much faster than in the previous case. Figure 1, Panel b displays the resulting numerical trajectory.

Recall now that the approach is to consider the problem as an iterative procedure, Therefore, instead of storing the complete information on the numerical trajectory, we store only the values of the map $M_n = M(v_n)$, for $n = 0, 1, \ldots$ where n is the number of the cycle. Figure 2 Panel a shows M_n as a function of n obtained for the computations with, $v_0 = 0$ and with $v_0 = \bar{v} = 0{\cdot}43333$. Note that starting with $v_0 = \bar{v}$, we are practically there after ≈ 10 cycles, as compared with starting with $v_0 = 0$ which requires some 50 cycles for convergence.

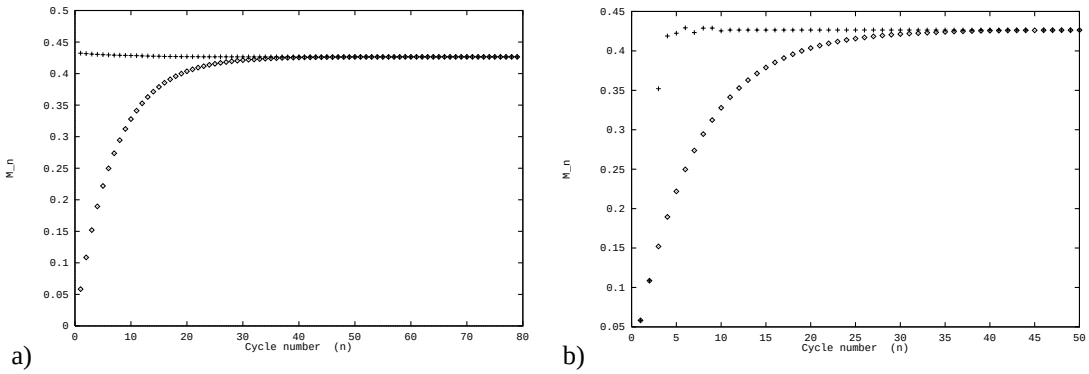

Figure 2. The iterations M_n, $n = 0, 1, \ldots$ where n is the number of the cycle. (a) The $\diamondsuit$ and the $+$ symbols indicate the solution for $v_0 = 0$ and $v_0 = \bar{v} = 0{\cdot}43333$, respectively. (b) The $\diamondsuit$ symbol indicates the solution with $v_0 = 0$. The $+$ symbol indicates the accelerated solution with $v_0 = 0$.

Accelerating the convergence by applying Aitken's method to the iterations can reduce the number of required cycles. To illustrate, Figure 2, Panel b shows the values of M_n as a function of n obtained for the computations with $v_0 = 0$ with and without acceleration. With acceleration, the required number of cycles is reduced to ≈ 10. Applying Aitken's method to the solution with the improved guess $v_0 = \bar{v}$ reduces the number of required cycles to ≈ 6.

4.1. An example with $m = 2$

For $m = 2$, we demonstrate our method with the system

$$\begin{aligned} \dot{v}(t) &= \varepsilon(\text{sign}\,(u(t) - \delta\cos(\omega t)) - \alpha_1 u(t) + b_1); \\ \dot{u}(t) &= \varepsilon(\text{sign}\,(v(t) - \delta\cos(\omega t)) - \alpha_2 v(t) + b_2). \end{aligned} \tag{8}$$

The parameter values used here are $\varepsilon = 0{\cdot}01$, $\delta = 1$, $\alpha_1 = 3$, $\alpha_2 = 4$, $b_1 = b_2 = 1$, and $\omega = 1$ (*i.e.*, $T = 2\pi$).

Solving the system of equations for the zero order approximation, namely $\int_0^T F_{1,2}(\bar{u}, \bar{v}, t)\,\mathrm{d}t = 0$ gives the solution $(\bar{u}, \bar{v}) = (0{\cdot}3250, 0{\cdot}4333)$. Figure 3, Panel a displays the numerical trajectory $(u(t), v(t))$ as a function of t, obtained with the initial conditions $v_0 = u_0 = 0$ and with the improved initial conditions $u_0 = 0{\cdot}3250$, $v_0 = 0{\cdot}4333$. Figure 3, Panel b displays the two dimensional iterations M_n^u, M_n^v, as a function of the number of the cycle, n. The advantage of starting with a better initial guess, by use of small-ε asymptotics, is evident. Panel c shows that applying an acceleration procedure also improves the convergence.

4.2. Extending the method to cases with an unknown period

The basic averaging idea is simple, and can be extended in several ways. For a system of equations $m \geq 2$, as we illustrated in the second example for $m = 2$, the increase in time saving can be substantial. Further, an extended version of the our technique can be applied to nonlinear (autonomous) problems where the period is not known apriori.

One example which we briefly illustrate here is the Van der Pol (VDP) oscillator $\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}$ where $\varepsilon > 0$ is a parameter (see [3, Chapter 11] for the application of the averaging method to this equation). We can apply our numerical procedure to find the underlying periodic solution of the VDP equation, although the period T is not known *a priori.*

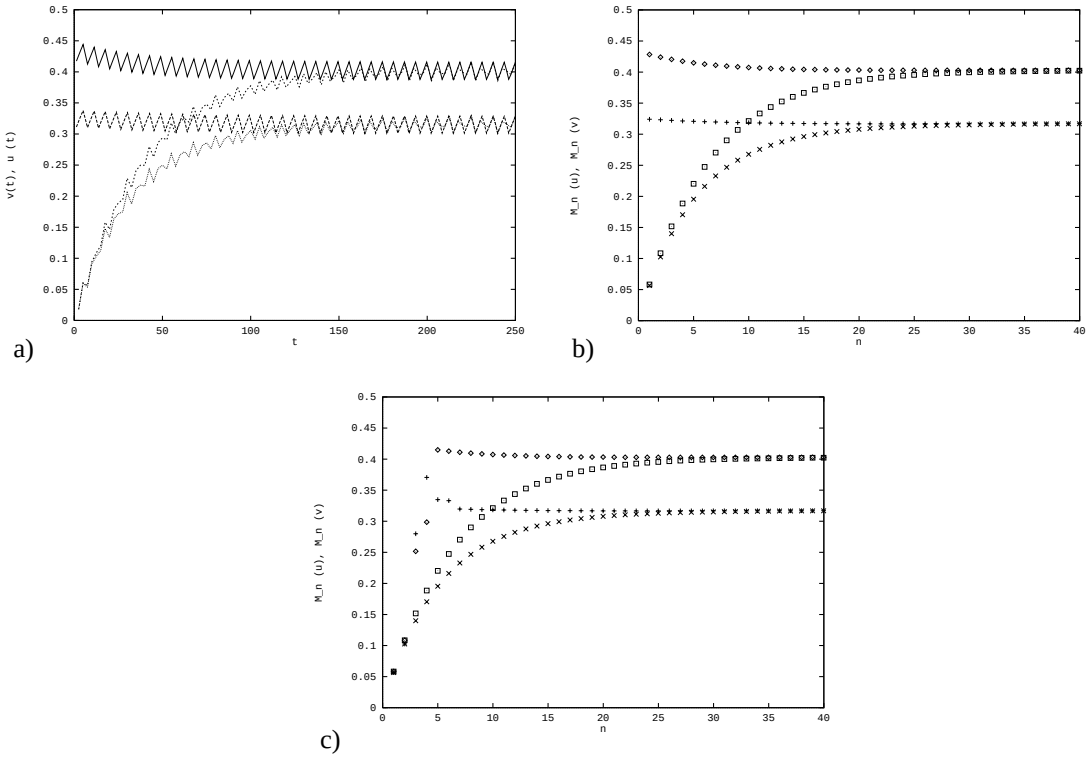

Figure 3. The solution of the system (8) using the Runge–Kutta method, $\mathrm{d}t = 0{\cdot}01256$. (a) The numerical trajectory $u(t)$ and $v(t)$ (+) for initial conditions $v_0 = u_0 = 0$ and for $u_0 = 0{\cdot}3250$, $v_0 = 0{\cdot}4333$. (b) The iterations M_n^u, M_n^v, $n = 0, 1, \ldots$ where n is the number of the cycle. The square and $\times$ symbols indicate the iterations for $v_0 = u_0 = 0$. The $\diamondsuit$ and the $+$ symbols indicate the iterations for $u_0 = 0{\cdot}3250$, $v_0 = 0{\cdot}4333$. (c) The accelerated solution with $v_0 = u_0 = 0$.

This can be done by considering T as an additional unknown and adding the requirement $\dot{x}(t_0) = \dot{x}(t_0 + T)$ to the procedure. The implementation uses a numerical minimization of $\mathcal{F}(x_0, \dot{x_0}, T) = (x(0) - x(T))^2 + (\dot{x}(0) - \dot{x}(T))^2$ where ε is a given parameter and $x(0) = x_0$, $\dot{x}(0) = \dot{x_0}$, T are the initial guess.

Computing a value of $\mathcal{F}$ requires the numerical integration of the VDP trajectory $x(s)$ over one period, starting from x_0 and $\dot{x_0}$. The numerical minimization procedure attempts to locate the limit cycle, by an appropriate choice of x_0 and $\dot{x_0}$. At the limit cycle, $\mathcal{F}$ is ideally zero. In practice, this is approximated by the values of x_0 $\dot{x_0}$, for which $\mathcal{F}$ is numerically minimized. Note that $\mathcal{F} = 0$ implies that $\int_0^T \dot{x}(s)\mathrm{d}s = 0$ and $\int_0^T \ddot{x}(s)\mathrm{d}s = 0$. We use these auxiliary conditions for validating the optimization.

For $\varepsilon = 0{\cdot}1$ and the initial guess $x(0) = 0$, $\dot{x}(0) = 1$, $T = 5$, this procedure produced the estimate $T \approx 6{\cdot}286$ to the unknown period, and enabled integrating the limit cycle. For comparison, the asymptotic expansion using the averaging method gives $T = 2\pi(1 + \frac{1}{16}\varepsilon^2) + O(\varepsilon^4) \approx 6{\cdot}28711$ ([3, Chapter 10]).

5. Discussion

We have demonstrated here the use of 'practical asymptotics' to a problem arising in industry (piezoelectric motors in our case). For a known qualitative limiting behavior of a limit cycle with a known period, we showed how finding the limit cycle, as a long time behavior, can be accelerated by a numerical acceleration technique, and also by a parameter expansion for the case of small oscillations. This idea, called also 'the averaging method', appears in the literature of dynamic systems in different contexts and applications. A comprehensive study can be found in [3, Chapters 4, 5, 11].

Acknowledgements

This research was supported by the U.S. - Israel Binational Science Foundation Grant number 97-00400, by a grant from the Israeli Ministry of Science, by the Technion Vice Provost for Research Funds - The David and Miraim Mondrey Research Fund, by The B. and G. Greenberg Research Fund (Ottawa), and by the Fund for the Promotion of Research of the University of Haifa.

References

1. B. Brogliato, *Nonsmooth Mechanics* (2nd ed.). Springer-Verlag: New York (1999) 400 pp.
2. O. Y. Zharii. Modeling of a mode conversion ultrasonic motor in the regime of slip. *IEEE Trasn. Ultrasonic, Ferroelectric Freq. Control* 40 (1993) 411–416.
3. J. K. Hale, *Ordinary Differential Equations* (2nd edition). Pure and Applied Mathematics series. New York: Wiley-Interscience (1980) 421 pp.
4. F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems.* Springer-Verlag: New York (1996) 303 pp.